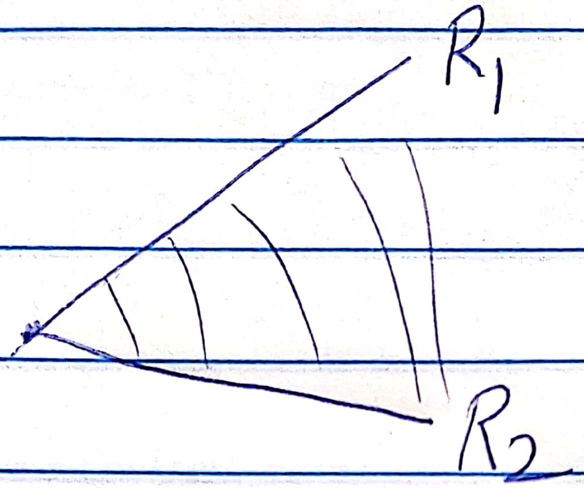
Fano Manifolds of Picard rank 2 with Simplest Contractions

— Supravat Sarkar

Let X be a smooth projective Fano variety over $\mathbb{C}$. We know that the Mori cone $\overline{NE}(X)$ is rational polyhedral, and every face can be contracted. For studying these contractions, the simplest nontrivial case is of $\rho(X) = 2$.

$\overline{NE}(X)$:



So, X has exactly 2 elementary contractions:

$$\begin{tikzcd} & X \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

So, X is built from $X_1, X_2, \varphi_1, \varphi_2$. We want to classify X, when the parts: $X_1, X_2, \varphi_1, \varphi_2$ are of simplest type.

Scanned with OKEN Scanner

[The simplest variety is $\mathbb{P}^n$. So, assume X_1, X_2 are projective spaces. What are the simplest type of elementary contractions?

Scanned with OKEN Scanner

1) Take a vector bundle $\begin{array}{c}\mathcal{E}\\ \big|\\ \mathbb{P}^n\end{array}$ of rank ≥ 2. Then $\begin{array}{c}\mathbb{P}(\mathcal{E})\\ \downarrow\\ \mathbb{P}^n\end{array}$ is an elementary contraction. (Projective bundle)

2) Take a smooth subvariety $W \hookrightarrow \mathbb{P}^n$ of codimension ≥ 2. Then $\begin{array}{c}Bl_W \mathbb{P}^n\\ \downarrow\\ \mathbb{P}^n\end{array}$ is an elementary contraction (Smooth blow up).

So, assume

$$\begin{tikzcd} X \arrow[r, "\varphi_2"'] \arrow[d, "\varphi_1"] & \mathbb{P}^m \\ \mathbb{P}^n & \end{tikzcd}$$

φ_1, φ_2 each are either ~~p~~projective bundle or smooth blow up.

<u>Aim:</u> Classify X.

Scanned with OKEN Scanner

So, there are 3 cases.

Case 1: φ_1, φ_2 both projective bundles.

Case 2: φ_1, φ_2 both smooth blow ups.

Case 3: φ_1 projective bundle, φ_2 smooth blow up.

Scanned with OKEN Scanner

$$\begin{tikzcd} X \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"] & \mathbb{P}^m \\ \mathbb{P}^n & \end{tikzcd}$$

Case 1: ~~both projective~~ bundles.

Theorem (Sato 1985):

Either $X = \mathbb{P}^n \times \mathbb{P}^m$

or $X = \mathbb{P}_{\mathbb{P}^n}(T_{\mathbb{P}^n})$

$\mathbb{P}_{\mathbb{P}^n}(T_{\mathbb{P}^n})$ is the flag variety

$X = \{ (x, H) \mid x \in \mathbb{P}^n, H \text{ hyperplane in } \mathbb{P}^n, x \in H \}$

$$\begin{tikzcd} X \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & (\mathbb{P}^n)^\vee \\ \mathbb{P}^n & \end{tikzcd} \qquad \begin{tikzcd} (x,H) \arrow[r, mapsto] \arrow[d, mapsto] & H \\ x & \end{tikzcd}$$

Both contractions identify X as $\mathbb{P}_{\mathbb{P}^n}(T_{\mathbb{P}^n})$.

Scanned with OKEN Scanner

Case 2:

$$\begin{tikzcd} & X \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Z_1 \hookrightarrow \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^n \hookleftarrow Z_2 \end{tikzcd}$$

φ a Cremona Transformation of $\mathbb{P}^n$.

Examples are less well-known.

Scanned with OKEN Scanner

Example: Say $\mathbb{P}^8 = \mathbb{P}(3\times 3 \text{ matrices})$

$$\mathbb{P}^8 \overset{\varphi}{\dashrightarrow} \mathbb{P}^8 \qquad A^* = (\text{cofactor matrix})^t$$

$$[A] \longmapsto [A^*] \qquad = \det A \cdot A^{-1} \text{ if } \det A \neq 0$$

$A^* = 0 \iff \operatorname{rk} A \leq 1$. So, indeterminacy locus of φ is the ~~Verones~~ Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$

$\varphi^{-1} = \varphi$, as $\overline{(A^{-1})}^{-1} = A$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & & \\ \mathbb{P}^8 & & \mathbb{P}^8 & \\ \mathbb{P}^2\times\mathbb{P}^2 \arrow[u, hook] & & \mathbb{P}^2\times\mathbb{P}^2 \arrow[u, hook] \end{tikzcd}$$

One can do the same thing with ~~skew~~ symmetric and skew symmetric matrices and get

$$\begin{tikzcd} & & X \arrow[dl] \arrow[dr] & & \\ \text{Veronese Surface} \arrow[r] & V \hookrightarrow \mathbb{P}^5 & & \mathbb{P}^5 \hookleftarrow V \end{tikzcd} \quad , \quad \begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathbb{P}^{14} & & \mathbb{P}^{14} \hookleftarrow Gr(2,6) \\ Gr(2,6) \arrow[u, hook] & & \end{tikzcd}$$

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Z_1 \hookrightarrow \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^n \hookleftarrow Z_2 \end{tikzcd}$$

Theorem (Crauder Katz 1991)

Suppose $\dim Z_1 = \dim Z_2 = m$. Let φ be given by homogeneous polynomials of deg c. Then one of the following holds; assuming Hartshorne's conjecture.

1. $c=2$, Z_1, Z_2 as in the examples + one exceptional case

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{Severi Varieties}}$

2. $c=3$, $n=3, m=1$: cubo-cubic transformation
   $n=7, m=4$
   $n=11, m=7$
   $n=15, m=10$
3. $c=4$, $n=9, m=6$
   $n=4, m=2$: Quadro-quadric transformation
4. $c=5$, $n=5, m=3$: Z_1, Z_2 are given by 5×5 minors of a 5×6 matrix of linear forms.

Hartshorne's conjecture: $Y \overset{\text{smooth}}{\hookrightarrow} \mathbb{P}^n$,

$\dim Y > \frac{2}{3}n \implies Y$ is a complete intersection

Scanned with OKEN Scanner

__Theorem (S. 2024):__ If $\dim Z_1 > \dim Z_2$, then φ is the quadro-cubic transformation

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
\mathbb{P}_E(\mathcal{E}) \hookrightarrow \mathbb{P}^4 & & \mathbb{P}^4 \hookleftarrow E
\end{tikzcd}$$

deg 5 elliptic scroll

E deg 5 elliptic ~~scroll~~ curve

These transformations have been classified

Scanned with OKEN Scanner

Case 3:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathbb{P}^{n+r} & W \arrow[l, hook'] \\ \mathbb{P}^n & & \end{tikzcd}$$

Example 1: W = linear subspace of dim $r-1$.

Scanned with OKEN Scanner

Example 2: Let $X = \{([A],[x]) \in \mathbb{P}^5 \times \mathbb{P}^2 \mid A_{2\times 3}, Ax = 0\}$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathbb{P}^5 \\ \mathbb{P}^2 & \end{tikzcd}$$

Fibres of π are $\mathbb{P}^3$,

φ is blow up along

$\{[A] \mid \operatorname{rk} A = 1\}$ = Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.

π is projectivization of $T(-1)^{\oplus 2}$.

Now restrict φ to a linear subspace H of $\mathbb{P}^5$ of codim k. $\varphi^{-1}H$ would be a projective bundle iff we have an ses of vector bundles

$$0 \to \mathcal{O}_{\mathbb{P}^2}^{k} \longrightarrow T(-1)^{\oplus 2} \longrightarrow E \to 0$$

such that $H = \mathbb{P}(H^0(E)) \subset \mathbb{P}(H^0(T(-1)^{\oplus 2}))$

For $0 \leq k \leq 2$ such ses are there

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{P}^4 & \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(1)) \arrow[l, hook'] \\ \mathbb{P}^2 & & \end{tikzcd}
\qquad
\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{P}^3 & \text{twisted cubic.} \arrow[l, hook'] \\ \mathbb{P}^2 & & \end{tikzcd}$$

Example 3: $\cancel{X = \{[A],[x]\}}$

$$X = \{([A],[x]) \in \mathbb{P}^9 \times \mathbb{P}^4 \mid A \text{ } 5\times 5 \text{ skew symmetric}, Ax = 0\}$$

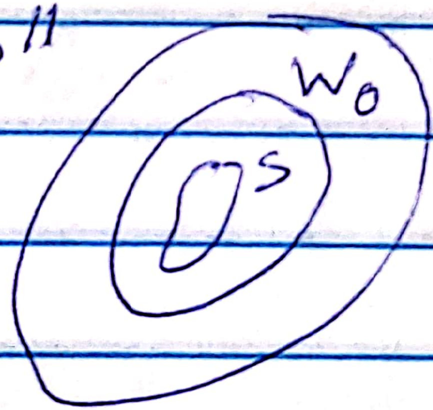
$$\begin{tikzcd}
\mathbb{P}(\Lambda^2 T(-1)) = X \arrow[r] \arrow[d] & \mathbb{P}^9 & Gr(2,5) \arrow[l, hook'] \\
\mathbb{P}^4 & &
\end{tikzcd}$$

Again can cut by $0 \le k \le 3$ hyperplanes.

Example 4:

$$\widetilde{X} = \{([A],[x]) \in \mathbb{P}^{11}\times\mathbb{P}^{3} \mid A_{3\times 4},\ Ax=0\}$$

$$\begin{tikzcd}
\mathbb{P}(T(-1)^3) = \widetilde{X} \arrow[r] \arrow[d] & \mathbb{P}^{11} \\
\mathbb{P}^3 &
\end{tikzcd}$$



W_0 : rk ≤ 2

S : rk ≤ 1

blow up along W_0.

Scanned with OKEN Scanner

Here W_0 is singular along S^5. But, we can cut by 6 hyperplanes to avoid S, hence getting smooth blow up. This is indeed possible: there are ses

$$0 \to \mathcal{O}_{\mathbb{P}^3}^6 \to T(-1)^3 \to E \to 0.$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{P}^5 & W \arrow[l, hook'] \\ \mathbb{P}^3 & & \end{tikzcd}$$

W a codim 2 subvar. of $\mathbb{P}^5$, not a complete intersection.

Scanned with OKEN Scanner

Theorem (S. 2024):

Assuming Hartshorne's conjecture, these are the only examples.

Scanned with OKEN Scanner

More refined form: Let $d \geq 2$ be such that

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbb{P}^{n+p} \arrow[dl, dashed] & W \arrow[l, hook'] \\ \mathbb{P}^n & & \end{tikzcd}$$

← given by deg d polynomials.

1) If $d=2$, X is as in example 2, 3

2) If $d \geq 3$, then either $d=3$, X is as in example 4, or $p = n/d$, W has same Betti numbers as $\mathbb{P}^{n-1}$

3) If Hartshorne's conjecture is true, the 2nd case in (2) is not possible.

Idea of proof:

$U = \mathcal{O}_{\mathbb{P}(E)}(1)$ $\qquad\qquad H_2$

$$\begin{tikzcd} E \subset X \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathbb{P}^{n+r} & W^m \arrow[l] \\ \mathbb{P}^n & & \end{tikzcd}$$

H_1

F_1 : line in fibre of π

F_2 : " " " " φ

	H_1	U	H_2	E
F_1	0	1	c	d
F_2	a	b	0	-1

Twist $\mathcal{E}$ by some $\mathcal{E}(\cdot m)$ to assume

$\mathcal{E}$ nef, $\mathcal{E}(-1)$ not nef. So, $0 \leq b < a$

Scanned with OKEN Scanner

$$\begin{pmatrix} H_1 \\ U \end{pmatrix} = \begin{pmatrix} \frac{da}{c} & -a \\ \frac{1+bd}{c} & -b \end{pmatrix} \begin{pmatrix} H_2 \\ E \end{pmatrix}$$

$$\Longrightarrow a = c, \; a \mid 1+bd$$

$$\Rightarrow H_2 = -bH_1 + aU. \quad \cancel{1}\, H_2$$

$I = H_2^{n+r} = (-bH_1 + aU)^{n+r}$ divisible by a

So, $a = 1$. $0 = b$.

Scanned with OKEN Scanner

Compute K_X in 2 ways and get:

$\operatorname{codim} W = \frac{n}{d} + 1$, $c_1(\mathcal{E}) = \frac{d-1}{d} n + 1$.

$b = 0 \Rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = U = H_2$,

so $\mathcal{E}$ is globally generated.

Scanned with OKEN Scanner

Case 1: $d=2$. So, $c_1 = \frac{n}{2}+1 = \operatorname{codim} W$

Claim: $\frac{n}{2} \leq r \leq \frac{n}{2}+3$.

Proof: $\frac{n}{2} \leq r$ follows from computation of the Betti numbers of X in 2 ways.

$\frac{n}{2}+3 \geq r$ follows from the result that W is not a complete intersection, and Hartshorne's Conjecture is proven for varieties defined by quadratic equations. $\square$

Scanned with OKEN Scanner

We explain $p = \frac{n}{2}$ case. Others are similar.
Computation of Betti numbers of X in 2 ways would show W has same Betti numbers as $\mathbb{P}^{n-1}$. Let $H^{2i}(W, \mathbb{Z})$ be gen. by H_2^i / l_i.

Scanned with OKEN Scanner

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{P}^{3n/2} \\ {\scriptstyle j}\uparrow & & \uparrow \\ E & & W^{n-1} \end{array}$$

Given an integral cohomology class α in W, $j_* \circ \varphi_E^*(1)^2 \circ (\varphi|_E)^*(\alpha)$ gives an integral cohomology class in X.

Decoding this description for $\alpha = H_2^i / \ell_i$, we get:

$$\ell_{k-i-1} \mid U^{k-i-1} (2U - H_1)^{i+1}.$$

This would give $\ell_1 = \ell_2 = \cdots = \ell_{\frac{n}{2}-1} = 1$.

This would give $l_1 = l_2 = \cdots = l_{\frac{n}{2}-1} = 1$.

By Poincare duality, $H_2^i / l_i \cdot H_2^{m-i} / l_{m-i} = H_2^m / e$.

Hence, $l_{\frac{n}{2}} = e$, where e is the degree of W.

n

Scanned with OKEN Scanner

So, $e \mid U^{\frac{n}{2}}(2U - H_1)$. Expand this out and

use $U^{\frac{n}{2}+1} = c_1 U^{\frac{n}{2}} H_1 + \cdots U^{\frac{n}{2}-1} H_1^2 \cdots$, $c_1 = \frac{n}{2} H$

to get $e \mid n+1$.

Scanned with OKEN Scanner

W can be shown to be nondegenerate. Now,

$\deg W = e \mid n+1$, $\operatorname{codim} W = c_1 = \frac{n}{2}+1$.

For any nondegenerate variety of degree e

and codimension c_1, we have $e \geq c_1 + 1$

This gives $e \geq \frac{n}{2}+2$, $e \mid n+1$. So, $e = n+1$.

Scanned with OKEN Scanner

In particular, $e = 2c_1 - 1$. Degree is still not too large compared to codimension. A result of Jinhyung Park classifies nondegenerate smooth projective varieties of degree small compared to codimension.

Scanned with OKEN Scanner

Their result:

Suppose $2 \leq c_1 \leq \dim W - 2$.

Then 1) $e \geq c_1 + 2$. Equality case is classified

2) If $e \geq c_1 + 3$, then $e \geq 2c_1 + 2$.

Scanned with OKEN Scanner

We have in our situation, $e = 2c_1 - 1$.

So, the above result would give a contradiction once we rule out small cases:

We need to guarantee: $2c_1 - 1 \geq c_1 + 3$, and

$$2 \leq c_1 = \frac{n}{2} + 1 \leq \dim W - 2 = n - 4.$$

This amounts to showing $n \geq 10$.

Scanned with OKEN Scanner

It remains to rule out cases for $n \leq 9$.

If $n \leq 9$, $c_1 \leq 5$. There is a classification of globally generated vector bundles $\mathcal{E}$ on $\mathbb{P}^n$ with $c_1(\mathcal{E}) \leq 5$. We check case by case, and get the examples we have seen as the only examples. $\square$

Scanned with OKEN Scanner